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MONOFRACTAL DENSITY FLUCTUATIONS AND SCALING LAWS FOR COUNT PROBABILITIES AND COMBINANTS

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ABSTRACT The relation of combinants to various statistics characterizing the fluctuation pattern of multihadron final states is discussed. Scaling laws are derived for count probabilities and combinants in the presence of homogeneous and clustered monofractal density fluctuations. It is argued that both types of scaling rules are well suited to signal Quark-Gluon Plasma formation in a second-order QCD phase transition.

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1. Introduction

There is a renewed interest in an almost forgotten statistic well suited to characterize the fluctuation pattern and the nature of correlations of multihadron final states. This statistic is the set of combinants introduced 15 years ago by Gyulassy and Kauffmann [1-2]. After their introduction the combinants received little attention, only three papers have been published concerning these quantities [3-5]. Probably this was caused by the fact that the existence of combinants requires nonzero probability for detecting no particles at all (and precise data for this probability) which was the exception rather than the rule 15 years ago in the analysis of multiplicity distributions. It is worth mentioning that the Buras-expansion of multiplicity distributions invented years earlier [6] also utilizes the combinants but the emphasis in this work was made on the factorial cumulant moments and the above important requirement was unnoticed. This expansion was applied in [7-8]. Three further publications investigating the combinants have been appeared quite recently [9-11]. The work of Szapudi and Szalay [10] is particularly recommended to the interested reader. Although not explicitly mentioned the combinants are present also in the work of Lupia, Giovannini and Ugoccioni [12].

This letter concentrates on the study of various scaling laws of count probabilities and combinants that appear in the presence of monofractal density fluctuations. The discussion of the relationship between combinants and the more conventional measures of fluctuations and correlations is also the goal of the paper. Section 2 summarizes the basic properties of the combinants and a scaling law is provided which follows from the validity of the Linked Pair Approximation (LPA). A similar scaling law can be derived for count probabilities if the observed events have monofractal structure. This result will be presented in section 3. In section 4 the two types of scaling rules are reinvestigated for a Poisson superposition of clusters. Our conclusions are in section 5.

2. Combinants, LPA and scaling

The generating function $\mathcal{G}(z)$ of the multiplicity distribution P_n is conveniently defined by

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} P_n z^n. \tag{2.1}$$

In the analysis of various count probability distributions occuring in nature such as the distribution of galaxy-, hadron- and photocounts the most successfully applied distribution functions are infinitely divisible (e.g. Poisson, Gaussian, Negative Binomial, Lognormal). For discrete distributions this family is known also as the compound Poisson [13]. A discrete distribution is said to be infinitely divisible if

its generating function has the property that for all $k > 0$ integer $\sqrt[k]{\mathcal{G}(z)}$ is again the generating function of a certain distribution [13]. $\mathcal{G}(z)$ satisfies this property if and only if $\mathcal{G}(1) = 1$ and

$$\ln \mathcal{G}(z) = \ln \mathcal{G}(0) + \sum_{q=1}^{\infty} C_q z^q \tag{2.2}$$

where $C_q \geq 0$ and

$$\sum_{q=1}^{\infty} C_q = -\ln \mathcal{G}(0) < \infty. \tag{2.3}$$

Hence the probability of detecting no particles, the so-called void probability, is $P_0 = \mathcal{G}(0) > 0$ for infinitely divisible distributions. The quantities C_q in eqs. (2.2-3) are the combinants first studied in detail by Gyulassy and Kauffmann [1-2]. We can get the C_q by successively differentiating the logarithmic generating function:

$$C_q = \frac{1}{q!} \frac{d^q}{dz^q} \ln \mathcal{G}(z) \Big|_{z=0} \tag{2.4}$$

and $\mathcal{G}(z)$ takes the form

$$\mathcal{G}(z) = \exp\left(\sum_{q=1}^{\infty} C_q (z^q - 1) \right) \tag{2.5}$$

for infinitely divisible distributions. It is worth to introduce the count probability ratios $\mathcal{P}_n = (P_n/P_0)$. These can be expressed in terms of combinants by the recursion [10]

$$\mathcal{P}_n = \frac{1}{n} \sum_{q=1}^{n} q \, C_q \mathcal{P}_{n-q}. \tag{2.6}$$

The $\mathcal{P}_n$ involve only a finite number of combinants but P_0 (hence each P_n) requires all the C_q as is seen from eq. (2.3). In terms of probabilities the combinants are found to be

$$C_q = \mathcal{P}_q - \frac{1}{q} \sum_{n=1}^{q-1} n \, C_n \mathcal{P}_{q-n}. \tag{2.7}$$

Essentially this expression was obtained also in ref. [12] in a different context. From eq. (2.7) two advantageous features of the combinants are immediately seen. First, they require the knowledge of only a *finite* number of probabilities P_n. In q-th order the $P_{\leq q}$ are required. Second, we need not know the probabilities themselves. The combinants follow directly from the unnormalized topological cross sections since they involve only *ratios* of probabilities. It is also seen from eq. (2.7) that in general the combinants can take negative values as well for $q \geq 2$. In this case

a necessary condition of the infinite divisibility of the multiplicity distribution P_n is not satisfied.

In the analysis of correlations in restricted domains of phase-space a basic set of quantities are the integrated cumulant correlation functions providing the factorial cumulant moments, f_q, of the underlying multiplicity distribution [14-15]. The factorial cumulants are defined by the Taylor expansion of the logarithmic generating function:

$$\ln \mathcal{G}(z) = \sum_{q=1}^{\infty} \frac{(z-1)^q}{q!}\, f_q. \tag{2.8}$$

One of the main advantages of the combinants is their close relationship to the factorial cumulant moments. This manifests itself e.g. in the very simple expressions connecting the two quantities. The factorial cumulants can be expressed in terms of combinants according to

$$f_q = q! \sum_{n=q}^{\infty} \binom{n}{q} C_n \tag{2.9}$$

and the combinants read as

$$C_n = \sum_{q=n}^{\infty} \binom{q}{n} \frac{(-1)^{q-n}}{q!}\, f_q \tag{2.10}$$

in terms of factorial cumulants [2,6,10]. They also share some important common features. For the Poisson distribution we have $C_1 = \bar{n}$ and $C_{\geq 2} = 0$. Hence nonvanishing higher-order combinants measure the degree of deviation from Poissonian behaviour. The combinants exhibit the additivity property of the cumulant moments. When a random variable is composed of statistically independent random variables the corresponding combinants are additive in those of the independent components [2].

In the last few years much interest has been devoted to the analysis of correlations in the framework of the Linked Pair Approximation [16-17]. In the LPA the normalized factorial cumulant moments, $K_q = f_q/\bar{n}^q$, obey the recurrence relation

$$K_q = A_q K_2^{q-1} \tag{2.11}$$

where the coefficients A_q should ideally be independent of reaction type, energy, binsize and phase-space dimension. In hadron-hadron collisions at $\sqrt{s} = 200 - 900$ GeV the validity of LPA was confirmed with constant $A_{3,4}$ over the investigated range of energy and binsize [18]. The large uncertainties of A_5 permit a clear

conclusion. At lower energies, $\sqrt{s} = 22$ GeV, the constancy of the A_q over binsizes was found to be violated by the NA22 Collaboration [19]. This may be caused by the absence of translation invariance. In nucleus-nucleus collisions the A_q are essentially zero indicating that only two-particle correlations are present [20].

Beyond the direct test of eq. (2.11) using the factorial cumulants there is another possibility to check the LPA scheme. It is provided by a scaling feature of the combinants [11]. As is seen from eq. (2.10) if the K_q obey the LPA relation the ratios $C_n/\bar{n}$ chosen at fixed n should scale to a function of $\bar{n}K_2$ only:

$$\frac{C_n}{\bar{n}} = \sum_{q=n}^{\infty} \binom{q}{n} \frac{(-1)^{q-n}}{q!} A_q \, (\bar{n}K_2)^{q-1} = \chi_n(\bar{n}K_2). \tag{2.12}$$

Increasing n the contribution of an increasing number of low-order factorial cumulant moments is excluded from the χ_n. Eq. (2.12) provides the generalization of the scaling law obtained for the void probability in the framework of the LPA [21,12]. As we shall see scaling combinants may appear also in the presence of monofractal density fluctuations.

3. Monofractals and scaling count probabilities

In recent years the observations revealed that in all types of reactions large nonstatistical fluctuations exist in the local particle density [22]. Varying the resolving power, i.e. the size of the phase-space domain in which the particles are counted the density irregularities resembled the scale-invariant, intermittent fluctuations that appear in a turbulent fluid [23-24]. This manifests itself in the power-law dependence of the normalized factorial moments on the resolution characterized by the anomalous dimensions d_q or intermittency exponents $(q-1)d_q$. The factorial moments, ξ_q, are defined by the Taylor expansion of $\mathcal{G}(z)$:

$$\mathcal{G}(z) = \sum_{q=0}^{\infty} \frac{(z-1)^q}{q!} \, \xi_q. \tag{3.1}$$

Measuring factorial moments in a certain phase-space domain is equivalent to measuring the underlying multiplicity distribution. Let us consider the relation between the ξ_q and the P_n. The factorial moments are expressed in terms of probabilities according to

$$\xi_q = q! \sum_{n=q}^{\infty} \binom{n}{q} P_n \tag{3.2}$$

where the summation produces the binomial moments. The count probabilities take the form

$$P_n = \sum_{q=n}^{\infty} \binom{q}{n} \frac{(-1)^{q-n}}{q!} \, \xi_q \tag{3.3}$$

when expressed in terms of factorial moments. Comparing eqs. (3.2-3) to eqs. (2.9-10) we see that the relationship between probabilities and factorial moments, generated by $\mathcal{G}(z)$ and $\mathcal{G}(1+z)$, is the *same* as the relationship between combinants and factorial cumulants, generated by $\ln \mathcal{G}(z)$ and $\ln \mathcal{G}(1+z)$. This important feature which is valid for ordinary moments and cumulants too was first emphasized by Kauffmann and Gyulassy [2].

The above correspondence enables us to apply the same scaling argument for count probabilities which was utilized in the previous section for combinants. Let us assume that instead of the normalized factorial cumulants, K_q, the normalized factorial moments $F_q = \xi_q/\bar{n}^q$ obey the LPA-type recurrence relation with constant coefficients A_q:

$$F_q = A_q F_2^{q-1}. \tag{3.4}$$

Eq. (3.4) is a special case of the famous Ochs-Wosiek empirical relation [25] that can be formulated as

$$F_q = A_q F_2^{(q-1)d_q/d_2}. \tag{3.5}$$

From currently available data for the factorial moments, $A_q \approx 1$ and the intermittency exponent ratios $(q-1)d_q/d_2$ are largely independent of reaction type, energy and phase-space dimension [22]. If F_2 changes according to a power-law with varying resolution eq. (3.4) corresponds to monofractal density fluctuations characterized by $d_q/d_2 = 1$, i.e. by the unique anomalous dimension d_2 given by the intermittency exponent of F_2. Plugging into eq. (3.3) with ξ_q expressed according to eq. (3.4) and evaluating the ratios $P_n/\bar{n}$ one arrives at

$$\frac{P_n}{\bar{n}} = \sum_{q=n}^{\infty} \binom{q}{n} \frac{(-1)^{q-n}}{q!} A_q \, (\bar{n}F_2)^{q-1} = \eta_n(\bar{n}F_2). \tag{3.6}$$

That is to say, if eq. (3.4) holds for the factorial moments the $P_n/\bar{n}$ chosen at fixed n not depend on reaction type, energy, binsize and dimension arbitrarily, but only through the momentum combination $\bar{n}F_2$. Picking up the P_n from the tail of the multiplicity distributions one subtracts the contribution of the low-order factorial moments from the scaling functions. In this manner we are able to go further in testing the monofractality of density fluctuations than the highest order factorial moment that can be extracted from observational data. Since monofractal patterns in multihadron final states are expected to form during a second-order QCD phase transition the collapse of the $P_n/\bar{n}$ to the scaling curves $\eta_n(\bar{n}F_2)$ as $d_q/d_2 \to 1$ may serve as a signature of Quark-Gluon Plasma formation. This point will be discussed in the next section. We note that the absence of translation invariance can lead to the violation of the binsize-independence of A_q and thus the scaling behaviour of P_n.

4. Scaling laws in the Poisson cluster model

Let us assume that the observed events are composed of identical groups of particles (clusters, clans, Quark-Gluon Plasma droplets or whatever) distributed according to a Poisson process. The generating function of the total event multiplicity distribution P_n becomes [26-27]

$$\mathcal{G}(z) = \exp\left(\bar{\mathcal{C}}(\mathcal{H}(z) - 1)\right) \tag{4.1}$$

which is the convolution of the Poissonian generating function of the distribution of clusters having mean $\bar{\mathcal{C}}$ and the generating function $\mathcal{H}(z)$ of the distribution of particles within a single cluster. Eq. (4.1) is another way of writing the generating function of infinitely divisible distributions, eq. (2.5), with $\bar{\mathcal{C}} = -\ln \mathcal{G}(0)$ and

$$\mathcal{H}(z) = 1 - \frac{\ln \mathcal{G}(z)}{\ln \mathcal{G}(0)} = \sum_{q=1}^{\infty} p_q z^q. \tag{4.2}$$

In eq. (4.2) the q-particle count probability in a cluster, p_q, is found to be [10-11]

$$p_q = \frac{C_q}{\sum_q C_q}. \tag{4.3}$$

Since $\mathcal{H}(0) = 0$ each cluster must contain at least one particle, i.e. $p_0 = C_0 = 0$. The two basic parameters of the model, the average cluster multiplicity, $\bar{\mathcal{C}}$, and the average multiplicity in a cluster, $\bar{q}$, are given by [28,12]

$$\bar{\mathcal{C}} = -\ln P_0 \qquad \text{and} \qquad \bar{q} = \bar{n}/\bar{\mathcal{C}}. \tag{4.4}$$

In the Poisson cluster picture the normalized factorial moments of a single cluster, $\mathcal{F}_q$, can be expressed in terms of $\bar{\mathcal{C}}$ and the normalized factorial cumulants K_q of the total events as [29-30]

$$\mathcal{F}_q = \bar{\mathcal{C}}^{q-1} K_q. \tag{4.5}$$

Writing K_q according to the LPA relation, eq. (2.11), yields [28,11]

$$\mathcal{F}_q = A_q \mathcal{F}_2^{q-1}. \tag{4.6}$$

Thus we see that a Poisson superposition of clusters which obey the Ochs-Wosiek relation, eq. (3.5), with intermittency exponent ratios $(q-1)$ leads to the validity of the LPA relation, eq. (2.11), for the total events. Comparing eqs. (2.3) and (4.3-5) one finds the following relationship between the scaling functions derived in the previous sections:

$$\chi_q(\bar{n}K_2)\Big|_{\text{total event}} = \eta_q(\bar{n}F_2)\Big|_{\text{single cluster}}. \tag{4.7}$$

The two sets of scaling functions appearing on the two different levels are equivalent. On the basis of eqs. (4.5-7) let us collect the correspondences between total event observables and quantities characterizing a single cluster:

i) If the events are Poisson superpositions of *selfsimilar* clusters we observe the power-law dependence of K_q instead of F_q (the F_q show a bending upward behaviour on log-log plot). Through the slopes of the K_q we actually see the slopes of the $\mathcal{F}_q$, that is, the fractal structure of a single cluster [29-30].

ii) When the Poisson clusters underlying the events have *monofractal* structure and the $\mathcal{F}_q$ obey eq. (3.4) the K_q exhibit the LPA relation, eq. (2.11). Thus the validity of LPA may arise as the consequence of randomly superimposed monofractal clusters [28,11].

iii) Accordingly, by the observation of χ_q-scaling for the total events confirming the validity of LPA one actually sees η_q-scaling for single clusters confirming their monofractal structure.

The search of monofractal patterns in multihadron final states plays a distinguished role in intermittency analyses. It has been conjectured that monofractal density fluctuations could signal a phase transition from the Quark-Gluon Plasma [31-32]. At the critical point of a second-order QGP phase transition to the hadron gas intermittent fluctuations are expected to occur characterized by a unique anomalous dimension as in the Ising model [33]. Białas and Hwa proposed the experimental confirmation of the validity of eq. (3.4) as a QGP signature, i.e. the q-independence of the anomalous dimensions d_q in eq. (3.5) [31]. Another proposal, put forward by Peschanski, is based on the description of the monofractal fluctuations at the critical point in terms of clusters [32]. The unique anomalous dimension is interpreted in this case as the fractal dimension of the random set of intermittent clusters of the ordered phase inside the disordered one at the transition. For each proposal one has to assume that the monofractal density fluctuations survive the further evolution of the system toward the hadronic final state. According to eq. (4.7) both sets of scaling functions could signal QGP formation. Events possessing homogeneous monofractal fluctuations exhibit η_q-scaling whereas events composed of random monofractal clusters give rise (through the η_q-scaling of the clusters) to χ_q-scaling for the observed events.

5. Conclusions

Over the last few years the analysis of fluctuations and correlations in restricted domains of phase-space revealed that the higher-order statistics characterizing the fluctuation pattern are frequently related to two-particle statistics in

a very simple manner [22]. One example is the LPA relation, eq. (2.11), for the factorial cumulant moments. The validity of LPA was confirmed for $p\bar{p}$ collisions at CERN collider energies with constant coefficients A_q close in magnitude to the Negative Binomial values $A_q = (q-1)!$ [18]. Another frequently encountered relationship between second- and higher-order statistics is the Ochs-Wosiek relation for the factorial moments. In addition to the simplicity of eq. (3.5) it has a kind of universality: rather independently of reaction type, energy, binsize and phase-space dimension, the coefficients $A_q \approx 1$ and the intermittency exponent ratios $(q-1)d_q/d_2$ are compatible with a Lévy-stable law with Lévy-index $\mu = 1.6$ [22].

Eq. (3.4), the Ochs-Wosiek relation with intermittency exponent ratios $(q-1)$ characterizes monofractal density fluctuations. Such patterns are expected to form during a second-order phase transition from the Quark-Gluon Plasma to the hadron gas [31-32]. Recent studies of the phase structure of QCD suggest that for two massless quark flavours QCD undergoes a second-order transition and this remains valid for three flavours provided that the strange quark mass is sufficiently large [34]. Hence it is of interest to find clear signatures of monofractal density fluctuations appearing in multihadron final states.

In this paper we have derived a scaling law for the count probabilities P_n in the presence of monofractal density fluctuations. It is expressed by eq. (3.6). According to this scaling law, if one plots the ratios $P_n/\bar{n}$ chosen at fixed n against the momentum combination $\bar{n}F_2$ the validity of eq. (3.4) with constant coefficients A_q results in a universal curve, $\eta_n(\bar{n}F_2)$, instead of many different behaviours corresponding to different reaction types, energies, binsizes and phase-space dimensions. Increasing n the low-order factorial moments can be excluded in a systematic manner from testing eq. (3.4). We have also found that a similar scaling law is satisfied for monofractal density fluctuations occuring in randomly distributed clusters. In this case the combinants show up a scaling behaviour: at fixed n the ratios $C_n/\bar{n}$ fall onto a universal curve when plotted against the momentum combination $\bar{n}K_2$. Through the scaling curves one observes the validity of eqs. (3.4) and (3.6) again but now for a single cluster. Both types of scaling rules provide easily measurable, clear signatures of homogeneous and clustered monofractal density fluctuations which could be the remnants of a second-order QCD phase transition. It will be interesting to see how these scaling predictions are satisfied in next heavy ion experiments.

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